Astronomy 121 Lab 1
Exploring Digital Sampling, Fourier Transforms, and the DSB Mixer

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ABSTRACT

We present a report on activities concerning radioastronomy bench measurements performed during the first 3 weeks of enrollment in Astronomy 121 at UC Berkeley. We performed several experiments to become acquainted with the equipment and signal processing involved in the Radio Astronomy Lab. This document focuses on four main areas: Fourier transforms, investigating the Nyquist frequency[1], investigating Dual Sideband Mixers, and phase detection using mixers. Each area depends heavily and contains information on Discreet Fourier Transforms, IDL programming & organization, and the UNIX operating system. This document strives to be an overview of each of these sections in context of the main areas.

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[1]Suggested by Harry Nyquist in a research paper, 1928

[2]Document created February 4, 2007

1. FOURIER TRANSFORMS & POWER SPECTRA

The field of signal analysis find its richness in the multiple representations that signals can take, with the transform between the time and frequency domains being the most fundamental. Fourier transforms take a signal representation from a time-basis to a frequency-basis through the use of orthogonality between basis vectors and a mathematical change of basis.[3]

The general Fourier transform integral is:

$$E(\nu) = \int E(t)e^{2\pi j\nu t}dt \qquad (1)$$

This equation in its discreet-time manifestation is what we use to analyze the discreet-time sampled signals:

$$X(\nu) = \sum X(n)e^{-2\pi j\nu n} \qquad (2)$$

Many algorithms exist that implement this change of basis vectors, most notable the Fast Fourier Transform. For this lab we use a pure mathematical rewriting of the Discreet Fourier Transform to calculate the frequency coefficients, and wrap this in a small IDL procedure[4] that allows us to take the DFT of any signal that we possess a time representation of, which consists of a time and a data array.

Notice that the DFT takes in a real signal spanning a real segment of time. It base-changes this to a series of complex values corresponding to the input signal's frequency content. Upon doing the inverse of this, we expect to find the original real signal again. This process can potentially be slightly complicated by the limited accuracy that computers afford us, and it is not uncommon to see machine error cause an imaginary part on the order of 10^{-09} or smaller on our reconstructed signals. It is acceptable to zero this miniscule imaginary component, which is what we did in the cases where an inverse transform was confirmed to return an imaginary part of this small order of magnitude.

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[3] A detailed analysis of Fourier transforms is beyond the scope of this paper. For more formation, see "The Structure and Interpretation of Signals and Systems" by *Lee & Varaiya.*

[4] Please see 6.1 for the IDL code

The Fourier transform representation itself is a series of complex exponentials, making them slightly difficult to visualize. What is more interesting is the power distribution over the frequency domain, which is what we can plot through finding the length of each complex exponential in the series. The power of each component is thus given by $\wp = |X(n)|^2$. This is the procedure we follow to create spectra of our data.

2. SAMPLING AND ALIASING

Every analysis we do with data captured from nature depends on the data being captured accurately and correctly. Without excellent data, science becomes impossible, and since our only data acquisition method is that of sampling, we thoroughly investigate the properties of digitally sampling analog signals. Sampling is the process by which an Analog to Digital converter gives the instantaneous digital value of an analog signal. This process is repeated at a certain frequency to build a digital approximation of an analog signal. We want to investigate the relationship between the sampling frequency and the signal frequency that allows for an accurate approximation.

Sampling at very high frequencies, except for the technical difficulties in building equipment that can handle it, causes bloated data files that contains much more information that we need for accurate analysis. Sampling at very low frequencies will clearly misrepresent fast signal. We hope to find a bounding number on the ratio of ν_{signal} to ν_{sample}

2.1. Experiments

We set up an iterative test to investigate the relationship between ν_{signal} and ν_{sample}. We picked a convenient sampling frequency $\nu_{sample} = 1.0 \times 10^6$ Hz and varied our signal frequency over the range $\nu_{signal} = (0.1, 0.2, ..., 0.9)$, taking 64 samples for each of the ν_{signal} values. We also used an oscilloscope to measure and record the approximate period of each sampled sine wave.

We expanded our dataset by doing two additional experiments to broaden our understanding of the extreme cases. For the first extreme case, we sampled at the same frequency as the input signal ($\nu_{signal} = \nu_{sample}$). For the second case, we sampled at a frequency much lower than the signal

frequency. ($\frac{\nu_{signal}}{\nu_{sample}} >> 1$)

2.2. Data & Data Analysis

We applied the Discreet Fourier Transform to each of the 9 ν_{signal} datasets, and plotted both the time and the frequency representations, visible in Figure 7. We can now correlate the known ν_{signal}, the recorded approximate periods, and the frequencies (ν_{dft}) of the Fourier transform peaks, visible in Table 1.

	ν_{signal} (Hz)	p_{signal} (s)	$\nu_{dft} Hz$
1	1.0×10^5	$100 \mu s$	1.0×10^5
2	2.0×10^5	$50 \mu s$	2.0×10^5
3	3.0×10^5	$34 \mu s$	3.0×10^5
4	4.0×10^5	$25 \mu s$	4.0×10^5
5	5.0×10^5	$20 \mu s$	5.0×10^5
6	6.0×10^5	$17 \mu s$	4.0×10^5
7	7.0×10^5	$14 \mu s$	3.0×10^5
8	8.0×10^5	$12 \mu s$	2.0×10^5
9	9.0×10^5	$11 \mu s$	1.0×10^5

Table 1: Analyzed Nyquist experiment data

As the signal frequency increases, the Fourier transform shows the power distribution moving to increasingly higher frequencies, until we reach $\frac{\nu_{signal}}{\nu_{sample}} = 0.5$. This is the point where we sample at double the frequency of the input signal. When we consider the time behavior of the sine wave input, we easily come to the conclusion that this means we take two samples per period. If we sample at the right points during the period, we get the peak and trough values of the sine wave, which is what we see in the plots.

A very interesting symmetry is revealed when we progress beyond this ratio value of $\frac{\nu_{signal}}{\nu_{sample}} = 0.5$. As Table 1 shows, our input frequency is still increasing linearly, but our DFT power spectrum plot shows that the power is concentrated around *lower* frequencies. In fact, the higher frequencies seems to be reflected around the ν_{signal} frequency where $\nu_{signal} = \frac{1}{2} \times \nu_{sample}$. Upon inspection of the Power Spectrum over an interval much larger than $\left(-\frac{1}{2} \times \nu_{sample}, +\frac{1}{2} \times \nu_{sample}\right)$, we find that the power spectrum actually repeats, and as ν_{signal} increases, the various peaks move in opposite directions, thus looking as if the signal's frequency content was reflected. This does not mean

that we can extract relevant information from signals that are sampled at a ratio $\frac{\nu_{signal}}{\nu_{sample}} > 0.5$, it simple mirrors the periodicity that the mathematics shows in the Fourier series coefficients.

We examine what happens when $\nu_{signal} = \nu_{sample}$) as our first extreme case. Since we are sampling only once per period, we expect to see all our samples to be the same value. This is indeed what we measure (see Figure 1), demonstrating that sampling at the same frequency as the signal gives us no useful information.
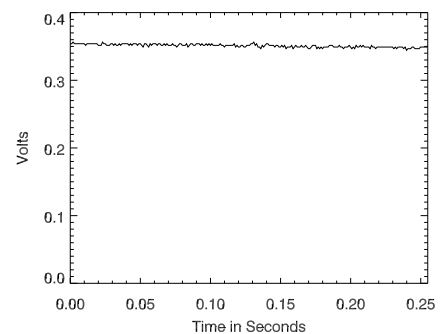

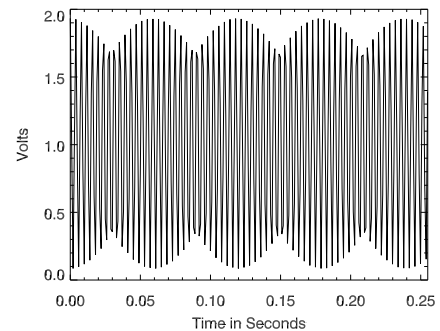Fig. 1.— 1kHz signal sampled at 1kHz

Fig. 2.— High frequency signal sampled at low frequency.

As our last extreme case, we sample at a frequency significantly lower than our signal frequency. In other words, we blatantly violate what our previous data shows us about sampling at a frequency such that $\nu_{signal} < \frac{1}{2} \times \nu_{sample}$. Collaborating with another group on collecting data, we

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decided to pick fairly arbitrary values, collect data on our own, and discuss the results. The signal frequency should go through a multitude of periods between single samples, thus we expect data that is in no way representative of the actual incoming waveform. Sampling a 10MHz signal at 11.7 kHz, our result in Figure 2 confirms our prediction.

2.3. Results & Discussion

We find that the minimum sampling frequency that allows for a perfect digital representation is bounded by a lower value that we call the Nyquist Criterion or Nyquist Frequency. The sampling rate should preferably be several times higher than the highest frequency present in a signal so that your data contains several points per period. None-the-less, we can now put a solid numerical bound on the ratio between the signal and sampling frequencies that will assure accurate digital representation of an analog signal:

$$\frac{\nu_{signal}}{\nu_{sample}} < \frac{1}{2} \tag{3}$$

3. DSB MIXERS

Our investigation of radioastronomy techniques now turn to *heterodyne mixing* and the Dual Sideband Mixer. The DSB mixer allows us to combine two signals into one through multiplication, and can best be understood through a review of sine & cosine mathematics.

The multiplication of two cosines can be expressed as:

$$E_s \cos \omega_1 \cos \omega_2 = \frac{E_s}{2} \left(\cos(\omega_1 - \omega_2) + \cos(\omega_1 + \omega_2) \right) \tag{4}$$

Thus, the multiplication of two waveforms through the mixer produces a waveform that contains both the sum and the difference of the two incoming waveforms. This seemingly innocuous property forms the basis of nearly every information transmission method in use today, since it allows us to manipulate the frequency of signals without losing their information. In radioastronomy we collect information from nature – specifically the sky. Almost always the information that we want is not at a frequency that we can readily sample at. For example, when we investigate the

21cm hydrogen line, we have an incoming signal at 1420 MHz, with all the information we want in a narrow band surrounding this frequency. It is practically impossible to feed that signal directly into a sampler since it is simply too high of a frequency. Mixers allow us to mix this signal with a local oscillator frequency, thereby shifting the band we are interested in down towards a practical frequency we can sample. It also allows us to transmit the information at a lower frequency, where attenuation in cables is less.

Thus, if we have a signal $\omega_0 \pm |\delta\omega|$ and a local oscillator of frequency ω_0, we can model the effect of the mixer through the following equation:

$$E_s \cos\left[(\omega_0 \pm |\delta\omega|)t\right] \cos\left[\omega_0 t\right] =$$
$$\frac{E_s}{2} \left(\cos\left[\pm|\delta\omega|t\right] + \cos\left[(2\omega_0 \pm |\delta\omega|)t\right] \right)$$

Thus, the slight offset $\delta\omega$ between the local oscillator and signal frequencies becomes shifted to be centered around the DC frequency. This makes for easy, painless sampling once the high frequency components are filtered out.

3.1. Experiments

We want to model the process of extracting a $\delta\omega$ signal from a $\omega_0 \pm |\delta\omega|$ signal by using a mixer. In practical terms, we would like to sample a signal that oscillates around a very high base frequency (say, the 21cm hydrogen line) and we need to use a mixer to shift this signal into our sampling range.

We use two frequency generators, one set at ω_0, the other at $\omega_0 \pm |\delta\omega|$. We take the two outputs and mix them together, taking the mixed signal into the oscilloscope and the ADC to sample it. Since we want to experiment with the process, we choose our frequencies such that we can sample all the frequency components in the output. We expect these to be $\pm|\delta\omega|$ and $2\omega_0 \pm |\delta\omega|$, thus we want our sampling frequency to be $\boldsymbol{\nu_{sample} > 2(2\omega_0 + |\delta\omega|)}$. We set our $\omega_0 = 1$MHz, and our $\delta\omega = 0.05$. We sampled at 10MHz, and separately recorded both $\nu_{signal} = \omega_0 + |\delta\omega|$ and $\nu_{signal} = \omega_0 - |\delta\omega|$. This means we have the complete output of the mixer ready for analysis.

3.2. Data Analysis

As always, we start with a Fourier analysis of the incoming signal. Figure 3 shows the time and frequency domain representation of our lower side-band mixed signal. It shows the sample plot and the power spectrum of $\nu_{signal} = \omega_0 - |\delta\omega|$. The upper sideband signal looks almost exactly the same. The signal looks like a fast-moving sine wave enveloped by a slow-moving sine wave, which corresponds to our mathematical model. When we study the frequency domain, we see that it has peaks at $\pm|\delta\omega|$ and at $2\omega_0 \pm |\delta\omega|$, exactly as predicted.

We now want to extract the lower signal. This corresponds to filtering out the high frequencies that we do not want to sample, concentrating on moving the information we need into the range of our equipment's capabilities in later labs. We achieve this by zeroing the high frequencies using IDL. We can then perform an inverse Fourier transform to create a signal containing only the low-frequency parts of the mixer's output - effectively extracting the information that was situated in proximity to the ν_{lo} frequency.

This process is slightly tricky, since the arrays we work with must stay symmetric to reproduce a real-valued signal. The input to this process consists of real-valued samples, and we expect to regenerate a real-valued signal. To ease the process, we used IDL's powerful "where" command that selects certain elements according to a given command. Figure 4 shows the final product of the low pass Fourier filtering (note the scale change!).
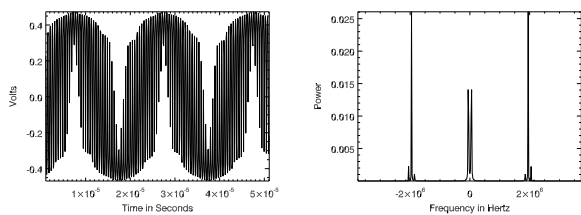

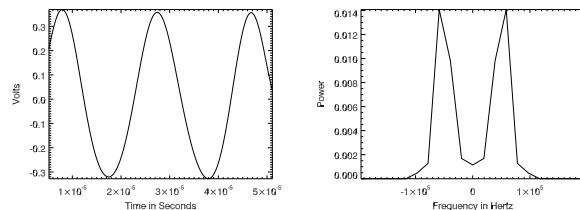Fig. 3.— Signal generated by mixing 1MHz and 1.05MHz, Sampling at 10MHz

Fig. 4.— Low-Pass Filtering a mixed 1MHz and 1.05MHz signal

3.3. Results & Discussion

Inspecting the filtered versus the original signal shows us the effects that mixers allow us to achieve. Keep in mind that the input to the mixer is two separate sine waves as you peruse the plots. We managed to successfully extract a 50kHz wave from a 1MHz modulated wave. This is exactly the process that the radio telescopes rely on to shift the high frequencies into more manageable frequency ranges, using hardware filters that accomplishes what we managed to do by hand in IDL.

4. DSB MIXER AS A PHASE DETECTOR

We used the mixer to experiment with two waves that are close together in the frequency domain, both with the same phase. When we use multiple radio telescopes connected together to gather data using interferometry, a similar situation occurs. The major deviation from our previous example is that all the radio telescopes in an interferometry setup are receiving at the same frequency - it is the phase that can differ between the telescopes as the earth rotates and causes a Doppler shift between telescopes.

We can expand the math to account for a constant difference between two signals of the same frequency:

$$\cos\left(at + \phi\right) \times \cos\left(at\right) = \frac{1}{2}\left(\cos\left(\phi\right) + \cos\left(2at + \phi\right)\right) \tag{5}$$

We also want to consider the case where there is only a slight offset in frequency between the two waves:

$$\cos\left(at+\phi\right)\times\cos\left(bt\right) \quad =$$
$$\frac{1}{2}\left(\cos\left((a-b)t+\phi\right)+\cos\left((a+b)t+\phi\right)\right)$$

4.1. Experiments

We set up the two frequency generators to generate sine waves at 1MHz. We then set the first frequency generator to a phase angle of 90 degree, and we increment the second frequency generator in steps of 5 degrees from 10 to 360 degrees, taking a DC voltage measurement for each step. Since we measure the DC voltage, we only log the constant part of the signal, corresponding to the $\cos\left(\phi\right)$. We plot this data over time.

To experiment with two signals that are slightly offset, we carry out the same procedure, except in this case we set the first frequency generator to 1MHz and the second frequency generator to 1MHz+1Hz, and sample as we increment the phase.

4.2. Data Analysis

Figure 5 shows how the DC voltage forms a sine wave as we vary the phase. Thus, the DC voltage corresponds to the relative phase difference between the two signals, which is also the $\cos\left(\phi\right)$ term in Equation 6.
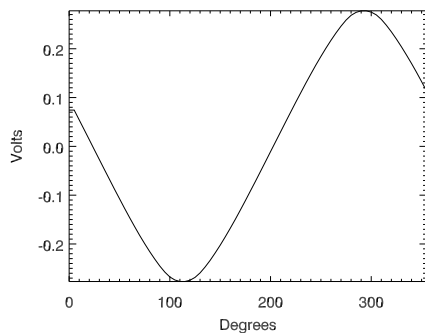

Fig. 5.— Two 1MHz signals, varying the phase over time.

Figure 6 shows the DC voltage plot over time as we vary the phase for the signal that mixes the 1MHz and 1MHz+1Hz signal. It looks like random noise, which is an accurate representation of the DC measurement in this case! Because of the one Hertz difference between the two signals, there is no pure DC in the output signal. The voltmeter thus picks up random time-average values of the low frequencies in the signal.
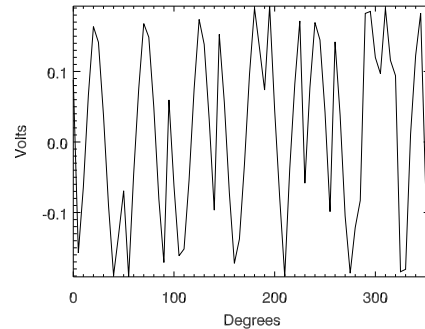

Fig. 6.— Mixed 1MHz and 1MHz+1Hz signal, varying the phase over time.

4.3. Results & Discussion

We showed that the phase between two same-frequency signals can be accurately described using the DC component of their mixed signal. This DC measurement of phase is exactly what interferometry depends on, and we expect to see this same technique in other areas of radioastronomy to find the phase difference between otherwise equal signals.

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5. CONCLUSION

This lab probed all the necessary parts of the field to gain an understanding of the underlying electronics, signal processing and programming that supports radioastronomy. We examined the most fundamental part of data analysis — the acquiring of data through digital sampling. Through our study of the ratio $\frac{\nu_{sample}}{\nu_{signal}}$ we found a strong lower limit on our sampling rate in relation to our signal. The relation $2\nu_{signal} < \nu_{sample}$, known as the Nyquist Criterion, is our basic guideline in our sampling of the data we want to analyze. We took this knowledge into our study of mixers, where we examined how to use mixers to shift frequencies into such ranges that they can be sampled by our equipment. The principles of this technique of *heterodyne mixing* was then applied to phase differences, and we found a way to measure the phase between two signals. The progression of our knowledge through these steps have led us to the point where we understand the concepts behind interferometary, and we can now head into serious radioastronomy observation.

6. APPENDIX

6.1. IDL Code

```
pro take_dft, fsample, nsample
;+
;NAME: TAKE_DFT, fsample, nsample
;PURPOSE:
;  Generate a sinusoid, feed it into the computer, and plot a time and frequency representation.
;INPUTS:
;  FSIGNAL, the frequency to generate
;  FSAMPLE, the sampling frequency
;  NSAMPLE, the sampling number
;-
;Create the time array
    time = findgen(nsample)
    time = time/fsample
    data = sampler(nsample, fsample)  ;Aquiring data
    plot, time, data, /xstyle, /ystyle
    print, ''
    print, 'Press s to save this sample, anything else to exit.'
    pressed_key=get_kbrd()

    case pressed_key of
        's': begin
            ;Create DTF power spectrum
            freq_a=(findgen(nsample)/nsample-0.5)*fsample
            dft, time, data, freq_a, fourier
            power=abs(fourier)^2

            filename1 = ''
            READ, filename1, PROMPT='Please enter an appropriatefilename'
            READ, fsignal, PROMPT='Please enter a Vsignal value: '

            !p.multi=[0,2,2]
            save, time, data, nsmaple, fsample, FILENAME='DFT_' + filename1 + '.sav'

            psopen, 'DFT_' + filename1 + '.ps', xsize=5, ysize=6, /inches
```

```
        plot, time, data, /xstyle, /ystyle,title='Sampled  V_sig='+$
string(fsignal,format='(%"%d")')+'Hz at V_smpl='+$
string(fsample,format='(%"%d")')+'Hz for '+$
string(nsample,format='(%"%d")'), xtitle='Time in seconds', ytitle='Volts'

        plot, freq_a, power, xtitle='Frequency in Hz',$
ytitle='Power', title='Spectrum of Sample Data at V_sig='+$
string(fsample,format='(%"%d")')+'Hz'
        psclose

    end
    else: print, 'Not saved...'

  endcase
print, 'Thank you, data capture completed!'
end
```
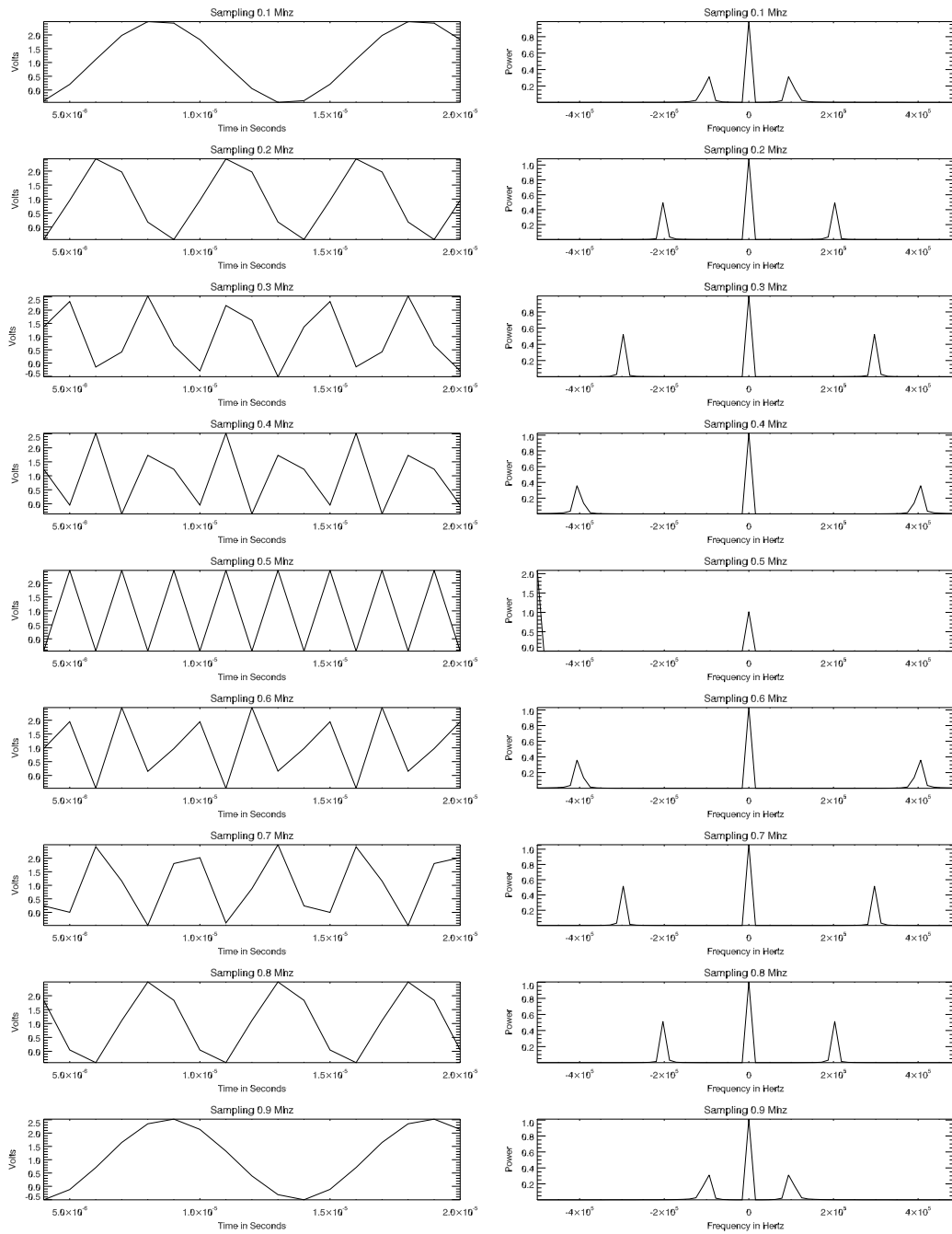
6.2. Nyquist Criterion experiment data

Fig. 7.— Nyquist Criterion experiment, Sampling at 1MHz